A Stabilized Lugannani-Rice Formula

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Abstract

The well-known approximate c.d.f formula proposed by Lugannani and Rice (1980) provides in many cases remarkably better accuracy than naive normal approximations. In extreme cases, however, it need not even be a probability. We propose a modification of the formula that is of the same asymptotic order of accuracy and similar typical performance, but is a stable approximation. That is, it is always a probability; and it is exact on an important family of arbitrarily skewed random variables.

Key Words: Saddlepoint methods; Inverse Gaussian variables; Stable approximations

I. Introduction:

The Lugannani and Rice (1980) expansion of the c.d.f. of a univariate asymptotically normal likelihood-based test statistic is widely admired for its simplicity and accuracy. (See e.g. Jensen 1975 Chapter 3). It seems to be so good and so well-known that perhaps the more interesting question is why, two decades later, it has not entered into ordinary statistical practice? Perusing Jensen's book suggests an explanation: he feels obligated to provide three quite distinct derivations. The one originally due to Lugannani and Rice is reasonably short; but it is a formal expansion of a complex path-integral from the Fourier inversion theorem. The connection with statistical concepts is at best quite obscure, at least to this reader. A second derivation involves a difficult approximate integration of the Daniels (1954) saddlepoint formula for the density of the test statistic. The third requires a change of variables in a Temme (1982) asymptotic expansion of the c.d.f.; which (as best this reader can tell) is a delicate translation of the complex analytic argument back into phase space. In sum, no one has yet explained to this working statistician why the formula is valid.

This note will attempt to provide a small step toward such an explanation. The Inverse Gaussian family of random variables (Schrödinger 1916) is a model for time-to-barrier of a randomly drifting system (how long will it take the drunk at a party to fall in the swimming pool?). It is a continuous, divisible, exponential family that is coming to be seen as of considerable practical as well as theoretical interest (Chhikara and Folks 1989). It has been noted as "truly remarkable" (Jensen 1995 p. 47) that the Daniels (1954) density formula is uniquely exact for this family. We will show here that the Inverse Gaussian family has another remarkable property: the Lugannani-Rice formula is transparently an approximation to its c.d.f. No other family with which the writer is familiar shares that property. This suggests a modification of the formula that shares its desirable simplicity and asymptotic properties, but which is exact for the Inverse Gaussian family. Finally we observe a connection between our new formula and the Woods, Booth, and Butler (1993) generalization of the Lugannani-Rice formula.

II. The Daniels Formula

Let a random variable X have density $f(x)$ and cumulant generating function $k(t) = \log\left[E\left(e^{Xt}\right)\right] = \log\left[m_X(t)\right]$. Then for fixed t let a tilted density be

$$f_t(x) = \frac{e^{xt} f(x)}{\int e^{Xt} f(X)\,\mathrm{d}X} = \frac{e^{xt} f(x)}{m_X(t)} = e^{xt - k(t)} f(x).$$

Note that f_t is a true density; its cumulant generating function is $k_t(s) = k(t+s) - k(t)$; and therefore its cumulants are $k_t^{(i)}(0) = k^{(i)}(t)$. Inverting the defining formula, we may write $f(x) = e^{k(t) - xt} f_t(x)$. Following Daniels (1954), we will choose t so that the tilted density has mean at the value of interest x; that is, solve $k'(t)=x$ for t. Then, because normal approximation tends to be most accurate near the mean, we will carry out a normal approximation to f_t in the usual way: its expectation is $\mu = x$; its variance is $\sigma^2 = k''(t)$; and we standardize by $Z = \frac{X_t - \mu}{\sigma}$. We will go further: imagining X to be an i.i.d. sum of n variates, we carry out a three-term Edgeworth expansion of the density of Z:

$$f_t^*(z) = \phi(z)\left\{1 + \frac{\kappa_3}{6}H_3(z) + \left[\frac{\kappa_3^2}{72}H_6(z) + \frac{\kappa_4}{24}H_3(z)\right]\right.$$

$$\left. + \left[\frac{\kappa_3^3}{1296}H_9(z) + \frac{\kappa_3\kappa_4}{144}H_7(z) + \frac{\kappa_5}{120}H_5(z)\right]\right\} + O(n^{-2})$$

where φ is the standard normal density; the cumulants of Z, the standardized cumulants, are $\kappa_l = \frac{k^{(l)}(t)}{k''(t)^{l/2}}$; and the $H_l(z)$ are the Hermite polynomials. For a general discussion of Edgeworth expansion, see e.g. Hall(1992).

Remembering that $\mu = k'(t) = x$, then to estimate the density at x we let $z = 0$:

$$f_t(x) = \frac{1}{\sigma}f_t^*\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sqrt{k''(t)}}f_t^*(0).$$

Now $H_l(z)$ is an odd polynomial for l odd; so that $H_{2l-1}(0) = 0$. In the other cases of interest, $H_4(0) = 3$ and $H_6(0) = -15$. Thus

$$f_t(x) = \frac{1}{\sqrt{k''(t)}}\left\{\frac{1}{\sqrt{2\pi}}\left[1 + \left(-\frac{5\kappa_3^2}{24} + \frac{\kappa_4}{8}\right)\right] + O(n^{-2})\right\}.$$

Inverse-tilting as above, we find

$$f(x) = \frac{e^{k(t)-xt}}{\sqrt{k''(t)}}\left\{\frac{1}{\sqrt{2\pi}}\left[1 + \left(-\frac{5\kappa_3^2}{24} + \frac{\kappa_4}{8}\right)\right] + O(n^{-2})\right\}.$$

This is the familiar Daniels density estimate with correction term of order $O(n^{-1})$.

Thinking of X as embedded in a natural exponential family (so that the natural parameter is t), we remember that the signed log-likelihood ratio statistic, or deviance, for that natural parameter is $z_l = \operatorname{sgn}(t)\sqrt{2[xt - k(t)]}$. Re-expressing the last result in terms of derivatives gives us a form that will be useful later:

Theorem 1: (Daniels)

$$f(x) = \frac{\phi(z_l)}{\sqrt{k''(t)}}\left(\left\{1 + \left[-\frac{5k'''(t)^2}{24k''(t)^3} + \frac{k^{iv}(t)}{8k''(t)^2}\right]\right\} + O(n^{-2})\right).$$

III. The Lugannani-Rice Formula.

There are of course more applications for tail-probabilities than there are for densities; numerical integration of the simple $O(n^{-1})$ Daniels formula is sometimes used to arrive at these. This is cumbersome and introduces quadrature error into the problem. Daniels

(1987) proposed formal integration of his Edgeworth expansion; and shows that this often works well in practice. But in 1980, Lugannani and Rice derived a much simpler formula for approximate tail areas also based on exponential tilting; and which in important cases has remarkably high accuracy. Their derivation, like Daniels (1954), uses complex contour integration; which though comparatively short and elegant, is somewhat opaque to this working statistician. I shall here, in the spirit of Jensen (1995), instead derive it by integration of our Daniels formula with $O(n^{-1})$ correction term. This will, in a subsequent section, be related to a formula with more concrete statistical implications.

We write

$$\mathrm{P}(X > x) = \int_x^\infty f(X)\mathrm{d}X = \frac{1}{\sqrt{2\pi}}\int_x^\infty \frac{e^{k(T)-XT}}{\sqrt{k''(T)}}\mathrm{d}X$$

$$+\frac{1}{\sqrt{2\pi}}\int_x^\infty \frac{e^{k(T)-XT}}{\sqrt{k''(T)}}\left\{\left[-\frac{5k'''(T)^2}{24k''(T)^3}+\frac{k^{iv}(T)}{8k''(T)^2}\right]\right\}\mathrm{d}X + O(n^{-2}).$$

We change variables by the tilting equation $X = k'(T)$ (which is monotone since the second derivative of k is a variance) to get $\mathrm{d}X = k''(T)\mathrm{d}T$. Then

$$\text{(1)} \qquad \mathrm{P}(X > x) = \frac{1}{\sqrt{2\pi}}\int_t^\infty \sqrt{k''(T)}\,e^{k(T)-Tk'(T)}\mathrm{d}T$$

$$+\frac{1}{\sqrt{2\pi}}\int_x^\infty \sqrt{k''(T)}\,e^{k(T)-Tk'(T)}\left\{\left[-\frac{5k'''(T)^2}{24k''(T)^3}+\frac{k^{iv}(T)}{8k''(T)^2}\right]\right\}\mathrm{d}T + O(n^{-2}).$$

We will refer to that second, correction, integral as E; and we shall evaluate that first, main integral.

Remember that $k(t) - tk'(t) = -z_l^2/2$; we will therefore carry out a further change of variables in that main integral $Z^2/2 = k'(T)T - k(T)$. Then $Z\mathrm{d}Z = Tk''(T)\mathrm{d}T$; we will addand subtract $\dfrac{\mathrm{d}Z}{\mathrm{d}T} = \dfrac{Tk''(T)}{Z}$ under the integral sign to get

$$\frac{1}{\sqrt{2\pi}}\int_t^\infty \sqrt{k''(T)}\,e^{k(T)-Tk'(T)}\mathrm{d}T = \frac{1}{\sqrt{2\pi}}\int_t^\infty Tk''(T)\left[\frac{1}{Z}+\frac{1}{T\sqrt{k''(T)}}-\frac{1}{Z}\right]e^{k(T)-Tk'(T)}\mathrm{d}T$$

$$= \frac{1}{\sqrt{2\pi}}\int_{z_l}^\infty e^{-Z^2/2}\mathrm{d}Z + \frac{1}{\sqrt{2\pi}}\int_t^\infty \left[\frac{1}{T\sqrt{k''(T)}}-\frac{1}{Z}\right]Tk''(T)e^{k(T)-Tk'(T)}\mathrm{d}T.$$

The first integral is a standard normal tail probability, which we will call $\Phi(z_l)$. We will integrate the second by parts $\mathrm{d}V = Tk''(T)e^{k(T)-xT}\mathrm{d}T$ and $U = \dfrac{1}{T\sqrt{k''(T)}} - \dfrac{1}{Z}$. Then $V = -e^{k(T)-Tk'(T)}$. The second integral becomes

$$\left[\frac{1}{t\sqrt{k''(t)}}-\frac{1}{z_l}\right]\frac{1}{\sqrt{2\pi}}e^{-z_l^2/2} + \frac{1}{\sqrt{2\pi}}\int_t^\infty e^{k(T)-Tk'(T)}\mathrm{d}U.$$

There is a further statistical interpretation to this expression. The t we found by tilting to mean x is the maximum likelihood estimate of the natural parameter in the exponential family in which we have embedded X. Wald (1943) noted that its variance may be estimated to be $1/k''(t)$, and its expectation is 0. Therefore the Wald statistic for testing our null distribution becomes $z_w = \frac{t-\mu}{\sigma} = t\sqrt{k''(t)}$. This appears in the integration above; we now know

$$(2) \qquad \frac{1}{\sqrt{2\pi}}\int_t^\infty \sqrt{k''(T)}\,e^{k(T)-Tk'(T)}\mathrm{d}T = \Phi(z_l) + \left[\frac{1}{z_w} - \frac{1}{z_l}\right]\phi(z_l) + \frac{1}{\sqrt{2\pi}}\int_t^\infty e^{k(T)-Tk'(T)}\mathrm{d}U .$$

To get a clearer idea of the size of that error integral, we will now approximate U; and to do that we need to express $1/Z$ in terms of T. We will begin by expanding k in a Taylor's series:

$$0 = k(0) = k(T) - Tk'(T) + \frac{T^2}{2}k''(T) - \frac{T^3}{6}k'''(T) + \frac{T^4}{24}k^{iv}(T) + \cdots \text{. Thus}$$

$$Z^2 = 2\left[k(T) - Tk'(T)\right] = T^2k''(T) - \frac{T^3}{3}k'''(T) + \frac{T^4}{12}k^{iv}(T) + \cdots$$

$$= T^2k''(T)\left[1 - \frac{T}{3}\frac{k'''(T)}{k''(T)} + \frac{T^2}{12}\frac{k^{iv}(T)}{k''(T)} + \cdots\right] \text{. Then}$$

$$\frac{1}{Z} = \frac{1}{T\sqrt{k''(T)}}\left[1 - \frac{T}{3}\frac{k'''(T)}{k''(T)} + \frac{T^2}{12}\frac{k^{iv}(T)}{k''(T)} + \cdots\right]^{-1/2} \text{ and applying the binomial theorem}$$

$$\frac{1}{Z} = \frac{1}{T\sqrt{k''(T)}}\left[1 + \frac{T}{6}\frac{k'''(T)}{k''(T)} - \frac{T^2}{24}\frac{k^{iv}(T)}{k''(T)} + \frac{T^2}{24}\frac{k'''(T)^2}{k''(T)^2} + \cdots\right] \text{. Now}$$

$$U = \frac{1}{T\sqrt{k''(T)}} - \frac{1}{Z} = -\frac{1}{6}\frac{k'''(T)}{k''(T)^{3/2}} + \frac{T}{24}\frac{k^{iv}(T)}{k''(T)^{3/2}} - \frac{T}{24}\frac{k'''(T)^2}{k''(T)^{5/2}} + \cdots .$$

We need its differential:

$$\frac{\mathrm{d}U}{\mathrm{d}T} = -\frac{1}{6}\frac{k^{iv}(T)}{k''(T)^{3/2}} + \frac{1}{4}\frac{k'''(T)^2}{k''(T)^{5/2}} + \frac{1}{24}\frac{k^{iv}(T)}{k''(T)^{3/2}} - \frac{1}{24}\frac{k'''(T)^2}{k''(T)^{5/2}} + \cdots$$

$$= -\frac{1}{8}\frac{k^{iv}(T)}{k''(T)^{3/2}} + \frac{5}{24}\frac{k'''(T)^2}{k''(T)^{5/2}} + \cdots .$$

Now we approximate the error term above:

$$(3) \qquad \frac{1}{\sqrt{2\pi}}\int_t^\infty e^{k(T)-Tk'(T)}\mathrm{d}U$$

$$= -\frac{1}{\sqrt{2\pi}}\int_x^\infty \sqrt{k''(T)}\,e^{k(T)-Tk'(T)}\left\{\left[-\frac{5k'''(T)^2}{24k''(T)^3} + \frac{k^{iv}(T)}{8k''(T)^2}\right]\right\}\mathrm{d}T + O\left(n^{-3/2}\right).$$

This expression is precisely the negative of the term we called E above. Combining the equations 1, 2, 3 above, we get

$$\mathrm{P}(X > x) = \Phi(z_l) + \left[\frac{1}{z_w} - \frac{1}{z_l}\right]\phi(z_l) + \mathrm{E} + O\left(n^{-2}\right) - \mathrm{E} + O\left(n^{-3/2}\right).$$

Theorem 2: (Lugannani and Rice) $\mathrm{P}(X > x) = \Phi(z_l) + \left[\frac{1}{z_w} - \frac{1}{z_l}\right]\phi(z_l) + O\left(n^{-3/2}\right)$.

IV. The Inverse Gaussian Family:

Schrödinger (1915) found that the time $Y > 0$ until a Brownian random process $X(y)$ with drift first encounters a given value $x \neq 0$ is a shape and scale family of continuous random variables that may be standardized to have density $f(y) = \frac{d}{\sqrt{2\pi}\, y^{3/2}} e^{-\frac{1}{2}\left(y^{1/2} - \frac{d}{y^{1/2}}\right)^2}$.

The connection to more usual parametrizations is given in Terrell (2003). The reader may verify as an exercise that the cumulant generating function is $k(t) = \log\left[\mathrm{E}\left(e^{tY}\right)\right] = d\left[1 - (1-2t)^{1/2}\right]$. Therefore $\mathrm{E}(Y) = d$ and $\mathrm{Var}(Y) = d$; and the family is asymptotically normal as $d \to \infty$.

We will pause to derive a well-known expression for its cumulative distribution function, as that will familiarize the reader with a standard technique for working with the density. If we could somehow replace part of the exponent in the density with a change of variables $Z = Y^{1/2} - d/Y^{1/2}$, then the exponent becomes that of a standard normal density. But where do we then obtain the necessary Jacobian $\mathrm{d}Z = \left(1/2Y^{1/2} + d/2Y^{3/2}\right)\mathrm{d}Y$? The second term is already present in our density, so it remains to obtain the first.

Perform a preliminary change of variables $W = d^2/Y$, to obtain the density $g(w) = \frac{1}{\sqrt{2\pi}\, w^{1/2}} e^{-\frac{1}{2}\left(w^{1/2} - \frac{d}{w^{1/2}}\right)^2}$. This density, for a new random variable that stands for the rate at which our Brownian motion first encounters the barrier, is slightly simpler than our original one, and has an equally interesting interpretation. It is often studied for its own sake. It can be seen to provide us with the first part of the Jacobian we need.

As is conventional in small-sample asymptotics, we will evaluate our cumulative distribution function from the right: we will find the tail probability $\mathrm{P}(Y > y)$. Let for the moment $y \geq d$ (so the time to the barrier is above average). Then

$\mathrm{P}(Y > y) = \int_y^\infty \frac{d}{\sqrt{2\pi}\, Y^{3/2}} e^{-\frac{1}{2}\left(Y^{1/2} - \frac{d}{Y^{1/2}}\right)^2} \mathrm{d}Y$ and for our inverse variable

$$\mathrm{P}(W > y) = \mathrm{P}\left(Y < d^2/y\right) = \int_y^\infty \frac{1}{\sqrt{2\pi}\, W^{1/2}} e^{-\frac{1}{2}\left(W^{1/2} - \frac{d}{W^{1/2}}\right)^2} \mathrm{d}W .$$

Averaging these, we get

$$\frac{1}{2}\left[\mathrm{P}(Y > y) + \mathrm{P}\left(Y < d^2/y\right)\right] = \int_y^\infty \frac{1}{\sqrt{2\pi}}\left(1/2Y^{1/2} + d/2Y^{3/2}\right) e^{-\frac{1}{2}\left(Y^{1/2} - \frac{d}{Y^{1/2}}\right)^2} \mathrm{d}Y .$$

The standard normal transformation above may now be invoked to get

$$(4)\quad \frac{1}{2}\left[\mathrm{P}(Y > y) + \mathrm{P}\left(Y < d^2/y\right)\right] = \int_{z_1}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}Z^2} \mathrm{d}Z = \mathrm{P}(Z > z_1) = \Phi(z_1)$$

where $z_1 = y^{1/2} - d/y^{1/2}$.

We need another relation between these two probabilities; look at the exponent in the density once again:

$$(5)\quad \left(Y^{1/2} - d/Y^{1/2}\right)^2 = Y - 2d + d^2/Y = -4d + \left(Y + 2d + d^2/Y\right) = -4d + \left(Y^{1/2} + d/Y^{1/2}\right)^2 .$$

Substituting in our integrals: $\mathrm{P}(Y > y) = e^{2d}\int_y^\infty \frac{d}{\sqrt{2\pi}Y^{3/2}} e^{-\frac{1}{2}\left(Y^{1/2}+\frac{d}{Y^{1/2}}\right)^2}\mathrm{d}Y$; and $\mathrm{P}(W > y) = \mathrm{P}\left(Y < d^2/y\right) = e^{2d}\int_y^\infty \frac{1}{\sqrt{2\pi}W^{1/2}} e^{-\frac{1}{2}\left(W^{1/2}+\frac{d}{W^{1/2}}\right)^2}\mathrm{d}W$. This suggests the possibility of a second transformation to standard normality: $Z = Y^{1/2} + d/Y^{1/2}$, which is monotone increasing for $Y \geq d$. Then $\mathrm{d}Z = \left(1/2Y^{1/2} - d/2Y^{3/2}\right)\mathrm{d}Y$. Half the difference of the two integrals becomes

$$\frac{1}{2}\left[\mathrm{P}\left(Y < d^2/y\right) - \mathrm{P}(Y > y)\right] = e^{2d}\int_y^\infty \frac{1}{\sqrt{2\pi}}\left(1/2Y^{1/2} - d/2Y^{3/2}\right) e^{-\frac{1}{2}\left(Y^{1/2}+\frac{d}{Y^{1/2}}\right)^2}\mathrm{d}Y .$$

Then our change of variables gets

$$(6)\quad \frac{1}{2}\left[\mathrm{P}\left(Y < d^2/y\right) - \mathrm{P}(Y > y)\right] = e^{2d}\int_{z_2}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}Z^2}\mathrm{d}Z = e^{2d}\Phi(z_2)$$

where $z_2 = y^{1/2} + d/y^{1/2}$. Subtracting (6) from (4) gets

Theorem 3: (Shuster 1968) Let Y be standard inverse Gaussian with mean and variance d. Then for $y \geq d$, $P(Y > y) = \Phi(z_1) - e^{2d}\Phi(z_2)$, where $z_1 = y^{1/2} - d/y^{1/2}$ and $z_2 = y^{1/2} + d/y^{1/2}$.

Finding cumulative probabilities for any $0 < u \leq d$ involves no further difficulties. Choose y so that $u = d^2/y$; then $y \geq d$. We sum expressions (4) and (6) to conclude

Corollary to proof: Let $0 < u \leq d$. Then $P(Y < u) = \Phi(z_1) + e^{2d}\Phi(z_2)$, where $z_1 = d/u^{1/2} - u^{1/2}$ and $z_2 = u^{1/2} + d/u^{1/2}$.

V. An Approximate Inverse Gaussian CDF

Notice that in the expressions for the distribution function, always $z_2 \geq 2\sqrt{d}$; so that for large shape parameter d it corresponds to the tail probability of a large standard normal deviate, which is very small. There is a familiar approximation to standard normal tail probabilities, $\phi(z)\left(\frac{1}{z} - \frac{1}{z^3}\right) < \Phi(z) < \phi(z)\frac{1}{z}$ for $z > 0$ (see e.g. Terrell 1999 p. 328 for the easy derivation). This provides the approximation $\Phi(z_2) \approx \phi(z_2)\frac{1}{z_2}$ for $4d >> 1$. In that case, for $y > d$ we have $P(Y > y) \approx \Phi(z_1) - \frac{e^{2d}}{z_2}\phi(z_2)$. Applying identity (5) to the exponents, we find

Theorem 4: For $y > d$, $P(Y > y) = \Phi(z_1) - \frac{1}{z_2}\phi(z_1) + O\left(d^{-3/2}\right)$; and for $y \leq d$ we have $P(Y \leq y) = 1 - \Phi(z_1) + \frac{1}{z_2}\phi(z_1) + O\left(d^{-3/2}\right)$.

One interpretation of this approximation is that for d large, and y fairly near the mean, we have found a normal approximation to Y by the (non-standard) transformation z_1.

VI. Saddlepoint Formulas for Inverse Gaussian Variables

Let us see how the saddlepoint formulae from sections II and III apply to the Inverse Gaussian variable Y of Section IV. Then as noted there the cumulant generating formula of this family is $k(t) = d\left[1 - (1-2t)^{1/2}\right]$. Then $k'(t) = \dfrac{d}{(1-2t)^{1/2}}$ and $k''(t) = \dfrac{d}{(1-2t)^{3/2}}$.

Solving $k'(t) = y$ we get $t = \dfrac{y^2 - d^2}{2y^2}$. Then using the notation $z_w = t\sqrt{k''(t)}$ from section III, we find $\dfrac{t}{z_w} = \dfrac{d}{y^{3/2}}$; and from section II,

$z_l = \operatorname{sgn}(t)\sqrt{2\left[tk'(t) - k(t)\right]} = y^{1/2} - d/y^{1/2} = z_1$. So the Daniels density formula is

$$f(y) \approx \frac{d}{y^{3/2}}\phi\left(y^{1/2} - d/y^{1/2}\right) = \frac{d}{\sqrt{2\pi}\,y^{3/2}}e^{-\frac{1}{2}\left(y^{1/2} - \frac{d}{y^{1/2}}\right)^2}.$$

We find that, for this particular family, the Daniels saddlepoint formula is exact! Blæsild and Jensen (1985) established that the Inverse Gaussian variables (and their normal limit) are the only cases for which this is true. This suggests that the formula is somehow connected to properties of the Inverse Gaussian family.

Now to work out the Lugannani-Rice approximation in the Inverse Gaussian case: $z_w = \dfrac{y^2 - d^2}{2yd^{1/2}}$. Then $\dfrac{1}{z_w} - \dfrac{1}{z_l} = -\dfrac{y^{1/2}}{y+d} = -\dfrac{1}{y^{1/2} + d/y^{1/2}}$. By the notation in section IV, let $z_2 = y^{1/2} + d/y^{1/2}$. Then the Inverse Gaussian Lugannani-Rice formula becomes $P(Y > y) \approx \Phi(z_1) - \frac{1}{z_2}\phi(z_1)$. Note that this is the simple approximation formula from section III. So even though L-R is not exact for the Inverse Gaussian family, it is a transparent approximation. This is not so for any other family with which the author is familiar.

VII. A General CDF Approximation

The results of the previous suggestion inspire us to work backwards: modify the Lugannani-Rice formula so that it will be exact in the Inverse Gaussian case, but still a good approximation in other cases. For a general random variable X, define z_m by the relation $\frac{1}{z_m} = \frac{1}{z_l} - \frac{1}{z_w}$ and infinite when the two statistics coincide. It therefore equals z_2 for the Inverse Gaussian case. Then the L-R formula becomes $P(X > x) \approx \Phi(z_l) - \frac{1}{z_m}\phi(z_l)$. For values of iid sums in the normal range [$z = O(1)$] it is established in section III that $z_l^2 = z_w^2 + O(n^{-1/2})$. It follows from the subsequent binomial expansion that $\dfrac{1}{z_m} = O(n^{-1/2})$.

In section IV it was noted that $\Phi(z_m) = \phi(z_m)\left[\frac{1}{z_m} - O(n^{-3/2})\right]$. Therefore

Theorem 5: $P(X > x) = \Phi(z_l) - \dfrac{\phi(z_l)}{\phi(z_m)}\Phi(z_m) + O(n^{-3/2})$ for $z_m > 0$; and

$P(X \le x) = 1 - \Phi(z_l) - \dfrac{\phi(z_l)}{\phi(z_m)}\left[1 - \Phi(z_m)\right] + O(n^{-3/2})$ for $z_m < 0$. Furthermore, the error is zero for the Inverse Gaussian family.

The formula may be slightly simplified to $P(X > x) \approx \Phi(z_l) - e^{\frac{1}{2}\left(z_m^2 - z_l^2\right)}\Phi(z_m)$. In approximation theory, a *stable* approximation is defined to be *the exact solution to a nearby problem.* Since in this case we may think of ourselves as using a method that will be exact anywhere in the arbitrarily skewed but asymptotically normal Inverse Gaussian family; then this is a stable approximation to any skewed random variable in the normal domain of attraction. We will therefore call it the Stable Lugannani-Rice (SL-R) formula. It is in general of the same order as the original method, and comparably simple.

VIII. Other Base Distributions

In 1993, Wood, Booth and Butler (WBB) proposed a generalization of the Lugannani-Rice formula that used other than normal probabilities in the approximation process. One of the cases they explored at some length was the case where the approximations used tail probabilities and densities from an Inverse Gaussian distribution. They found that this new formula worked in some cases where the usual Lugannani-Rice formula was particularly poor (see also Booth and Wood 1995). It will not be surprising from our results in this paper that there is a notable connection between the two formulae.

The WBB derivation followed the path-integral inversion of the characteristic function that was used in the original (1980) derivation of the Lugannani-Rice formula (for a clear exposition see Daniels 1987). Once again, though, there is an elementary derivation that parallels the results of section III. We begin with the Daniels (1954) density estimate of section II: let a random variable X (sometimes interpreted as a sum of n i.i.d variates) have density $f(x)$ and c.g.f. $k(t)$. Then $f(x) = \dfrac{e^{k(t) - tk'(t)}}{\sqrt{2\pi}\sqrt{k''(t)}}\left[1 + O(n^{-1})\right]$ where $x = k'(t)$.

Then we integrate it with that same change of variables

$$P(X > x) = \int_x^\infty f(X)dX = \frac{1}{\sqrt{2\pi}}\int_t^\infty \sqrt{k''(T)}\,e^{k(T) - Tk'(T)}\mathrm{d}T\left[1 + O(n^{-1})\right].$$

In section III, we performed another change of variables to make that exponent into that of a standard normal density. Instead, let us transform it into the corresponding part of another "base" density. Choose a random variable Y with density $h(y)$, usually from a family with a convenient variety of shapes similar to the one we wish to approximate and for which the computation of tail probabilities is readily available. We will further assume that the family is asymptotically normal with "sample size" (shape parameter) m. This was true of the two examples WBB investigated in detail. Then just as above we may write

$$P(Y > y) = \int_x^\infty h(Y)dY = \frac{1}{\sqrt{2\pi}}\int_s^\infty \sqrt{g''(S)}\,e^{g(S) - Sg'(S)}\mathrm{d}S\left[1 + O(m^{-1})\right]$$

where $g'(s) = y$. To transform the problem above into this more computationally tractable form, we make the change of variables $Sg'(S) - g(S) = Tk'(T) - k(T)$, where we choose consistent signs for S and T, since both sides achieve a simple minimum at 0. We will do so by adding and subtracting the Jacobian $\mathrm{d}S = \dfrac{Tk''(T)}{Sg''(S)}\mathrm{d}T$ in our tail integral

$$P(X > x) = \frac{1}{\sqrt{2\pi}}\int_s^\infty \sqrt{g''(S)}\,e^{g(S) - Sg'(S)}\mathrm{d}S$$

$$+\frac{1}{\sqrt{2\pi}}\int_t^\infty \left(\frac{1}{T\sqrt{k''(T)}} - \frac{1}{S\sqrt{g''(S)}}\right)Tk''(T)e^{k(T) - Tk'(T)}\mathrm{d}T + O(n^{-1})$$

where in the second integral S is thought of as a function of T.

Note that the various limits of integration were determined as follows: Given a lower tail limit x, determine t by solving $x = k'(t)$; then determine s by solving $sg'(s) - g(s) = tk'(t) - k(t)$ and giving them the same sign; and finally calculate $g'(s) = y$.

Then that first integral is $P(Y>y)+O(m^{-1})$. We carry out the second integration by parts $U=\frac{1}{T\sqrt{k''(T)}}-\frac{1}{S\sqrt{g''(S)}}$ and $\mathrm{d}V=Tk''(T)e^{k(T)-Tk'(T)}\mathrm{d}T$. Then $V=-e^{k(T)-Tk'(T)}$. Then

$$\frac{1}{\sqrt{2\pi}}\int_t^\infty\left(\frac{1}{T\sqrt{k''(T)}}-\frac{1}{S\sqrt{g''(S)}}\right)Tk''(T)e^{k(T)-Tk'(T)}\mathrm{d}T$$

$$=\sqrt{g''(s)}\left(\frac{1}{t\sqrt{k''(t)}}-\frac{1}{s\sqrt{g''(s)}}\right)\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{g''(s)}}e^{g(s)-sg'(s)}+\frac{1}{\sqrt{2\pi}}\int_t^\infty e^{k(T)-Tk'(T)}\mathrm{d}U\,.$$

The equivalent of the Taylor's series argument from section III establishes that the second integral is $O(m^{-1})$. The other term contains an approximation to the transformed density $h(y)$ to the same order accuracy, We find ourselves with

Theorem 6: (Wood, Booth, and Butler)

$$P(X>x)=P(Y>y)+\left(\frac{\sqrt{g''(s)}}{t\sqrt{k''(t)}}-\frac{1}{s}\right)h(y)+O(n^{-1},m^{-1})\,.$$

In case Y is normal, this becomes the Lugannani-Rice formula, where the order in m terms disappear and we have noted that the terms in n cancel to this order.

It remains to establish a connection with the stabilized Lugannani-Rice approximation of section VII. We noted earlier that WBB explored examples of Y which included the gamma family and the Inverse Gaussian family, which had arbitrary shape parameters corresponding to the m in our analysis. We noted earlier that $t\sqrt{k''(t)}$ is the approximately standard normal Wald statistic for the observed value x; therefore $s\sqrt{g''(s)}$ is a sort of Wald statistic for the transformed problem. It may be checked from the Taylor's expansion above that they are equal to $O(m^{-1/2})$. In the applications studied in WBB, one was free to choose the parameter m corresponding to the observed local skewness in X. It may be conjectured that in many cases that may be done so as to make the two Wald statistics precisely equal, leading to a collapsed local approximation $P(X>x)=P(Y>y)+O(n^{-1})$.

We will now observe that this is precisely what happens if Y is taken to be Inverse Gaussian with shape parameter $m = d$ from section IV: After a little algebra we find

$$\frac{1}{s\sqrt{g''(s)}}=\frac{1}{\sqrt{2[tk'(t)-k(t)]}}-\frac{1}{\sqrt{2[tk'(t)-k(t)]+4d}}=\frac{1}{z_l}-\frac{1}{z_m}\,.$$

Then the identity $\frac{1}{z_m}=\frac{1}{z_l}-\frac{1}{z_w}$ in our stabilized Lugannani-Rice formula is seen to correspond to choosing the shape parameter by the identity $z_m^2-z_l^2=4d$. Thus the Stabilized L-R formula is the special case of our collapsed WBB formula where the base distribution is Inverse Gaussian. Notice that this case is particularly salubrious: the Daniels density approximation which we integrate is exact, so there is no $O(m^{-1})$ error term; and the analysis that let to the SL-R formula established that the $O(n^{-1})$ term disappears as well. Furthermore, we established in section IV that its tail probabilities could be computed using standard normal tail probabilities; and finally, the value of y we need may be calculated from the usual test statistics in closed form. We interpret this as evidence that in a certain sense, under regularity conditions, likelihoods are asymptotically Inverse Gaussian.

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