Optimal stopping of a risk process with interest rates.

Bogdan Krzysztof Muciek*

Abstract

The following problem in risk theory is considered. An insurance company receives premiums and pays out claims that occur according to a renewal process. The capital of the company is invested at interest rate $\alpha \in [0, 1]$, claims increase at rate $\beta \in [0, 1]$. The aim is to find the stopping time that maximizes the capital of the company. The improvement to the previous models is presented: claims are paid immediately instead of at the end the investment period.

Keywords: Risk reserve process, optimal stopping, dynamic programming, interest rates

1 Introduction

The following problem in risk theory is considered. An insurance company, endowed with an initial capital $a \geqslant 0$, receives premiums and pays out claims that occur according to a renewal process $\{N(t), t \geqslant 0\}$, where $N(t)$ is the number of losses up till time t. Let T_i denote the time of the ith loss, then random variables $S_i = T_i - T_{i-1}$ are independent and identically distributed (i.i.d.) with cumulative distribution function (cdf) F; $T_0 = 0$. Let $X_1, X_2, \dots$ be a sequence of i.i.d. random variables with cdf H, which represents the discounted value of successive claims.

A typical process of capital assets of an insurance company is given by the formula

$$\bar{U}_t = a + ct - \sum_{n=0}^{N(t)} X_n, \tag{1}$$

where $a > 0$ is the initial capital, $c > 0$ is a constant rate of income from the insurance premiums, $X_0 = 0$ and $N(0) = 0$. Some interesting properties of the model given by (1) have been considered by E. Ferenstein & A. Sierociński (1997). U. Jensen (1997) investigated a similar model with a claim process modulated by periodic Markovian processes. The models mentioned above do not take inflation into consideration, which is a serious omission.

Muciek (2002) generalized the model considered by E. Ferenstein & A. Sierociński (1997) by introducing interest rates the premiums were invested with. Similarly value of claims were increasing (for example as a result of inflation). More precisely, the initial capital of the insurance company and received premiums were invested at a constant rate of return $\alpha \in [0, 1]$. Value of claims was increasing at rate $\beta \in [0, 1]$. In that case the process of capital assets of the insurance company was

$$\tilde{U}_t = ae^{\alpha t} + \int_0^t ce^{\alpha(t-s)} ds - \sum_{n=0}^{N(t)} X_n e^{\beta T_n}, \tag{2}$$

A similar model with the claim process modulated by periodic Markovian processes has been investigated by Schöttl (1998). Schöttl has used a smooth semimartingale representation to solve the optimal stopping problem. Muciek used dynamic programming to solve the model.

However models investigated by Muciek and Schöttl are still far away from the real economy. In fact both models were constructed assuming the claims are paid at the end of the investment period which in fact is not

*Wrocław University of Technology, Institute of Mathematics, Wybrzeże Wyspiańskiego 27, Wrocław, Poland, email: B.Muciek@im.pwr.wroc.pl; the research was supported by KBN grant no 2 P03A 021 22 (350228).

the case. The claims should be paid immediately and thus should decrease the invested capital. Such a situation is modeled by the following process

$$U_t = ae^{\alpha t} + \int_0^t ce^{\alpha(t-s)}ds - \sum_{n=0}^{N(t)} X_n e^{\beta T_n} e^{\alpha(t-T_n)}, \tag{3}$$

which reduces to

$$U_t = ae^{\alpha t} + \int_0^t ce^{\alpha(t-s)}ds - e^{\alpha t}\sum_{n=0}^{N(t)} X_n e^{\beta_1 T_n}, \tag{4}$$

where $\beta_1 = \beta - \alpha$. The equation (3) differs from (2) by the discount factor $e^{\alpha(t-T_n)}$. This change will have an impact on the form of the dynamic programming equations.

This paper presents the improvement to the solution presented by Muciek (2002). The notation and the technique used are the same. The form of optimal stopping is the same. Some calculations have changed and the final formulas (including the strong generator) have changed.

The return at time t is $\{Z(t), t \geqslant 0\}$:

$$Z(t) = \begin{cases} g_1(U_t)\mathrm{I}_{\{U_s>0, s\leqslant t\}} & \text{if } t \leqslant t_0, \\ 0 & \text{if } t > t_0, \end{cases} \tag{5}$$

where g_1 is a utility function.

For simplicity of notation, let $g(u,t) = g_1(u)\mathrm{I}_{\{t\geqslant 0\}}$. Then

$$Z(t) = g(U_t, t_0 - t)\prod_{j=0}^{N(t)} \mathrm{I}_{\{U_{T_j}>0\}}.$$

2 The optimization problem

In this section we will define an optimization problem for the model introduced in the previous section. This optimization problem will be solved in the next section.

Let $\mathcal{F}(t) = \sigma(U_s, s \leqslant t) = \sigma(X_1, T_1, \ldots, X_{N(t)}, T_{N(t)})$ be the σ-field generated by all the events up to time $t \geqslant 0$. Let $\mathcal{T}$ be the set of all stopping times with respect to the family $\{\mathcal{F}(t), t \geqslant 0\}$. Furthermore, for fixed K and for $n = 0, 1, \ldots, k < K$ let $\mathcal{T}_{n,K}$ denote the subset of $\mathcal{T}$, such that $\tau \in \mathcal{T}_{n,K}$ if and only if $T_n \leqslant \tau \leqslant T_K$ a.s.

Let $\mathcal{F}_n = \mathcal{F}(T_n)$. The essence of the considerations in the next chapter will be to find the optimal stopping time τ_K^*, such that

$$\mathsf{E}Z(\tau_K^*) = \sup\{\mathsf{E}Z(\tau) :\ \tau \in \mathcal{T}_{0,K}\}. \tag{6}$$

In order to find the optimal stopping time τ_K^*, we first consider optimal stopping times $\tau_{n,K}^*$, such that

$$\mathsf{E}(Z(\tau_{n,K}^*)|\mathcal{F}_n) = \operatorname{ess\,sup}\{\mathsf{E}(Z(\tau)|\mathcal{F}_n) :\ \tau \in \mathcal{T}_{n,K}\} \tag{7}$$

and using backward induction as in dynamic programming, we will obtain $\tau_K^* = \tau_{0,K}^*$

After finding the optimal stopping time τ_K^* for fixed K we will deal with unlimited number of claims and the aim will be to find the optimal stopping time τ^*, such that

$$\mathsf{E}Z(\tau^*) = \sup\{\mathsf{E}Z(\tau) :\ \tau \in \mathcal{T}\} \tag{8}$$

is fulfilled. It will be shown that τ^* can be defined as the limit of the finite horizon optimal stopping times.

3 Case with fixed number of claims

In this section we find the form of optimal stopping time in the finite horizon case, which means the optimal stopping time in the class $\mathcal{T}_{0,K}$, where K is finite and fixed (the number of claims is fixed, but the time of the Kth claim, ie. time horizon, remains nondeterministic). First we present dynamic programming equations satisfying

$$\Gamma_{n,K} = \operatorname{ess\,sup}\{\mathsf{E}(Z(\tau)|\mathcal{F}_n) : \tau \in \mathcal{T}_{n,K}\}, \qquad n = K, K-1, \ldots, 1.$$

Then in Corollary 3.4 we find optimal stopping times $\tau^*_{n,K}$ and τ^*_K and optimal mean values of return related to them.

The following representation lemma (see for example Davis(1976)) plays the crucial role in consequent considerations:

Lemma 3.1 *If $\tau \in \mathcal{T}_{n,K}$, there exists a positive, $\mathcal{F}_n$-measurable random variable ξ, such that*

$$\tau \wedge T_{n+1} = (T_n + \xi) \wedge T_{n+1} \qquad a.s.$$

Let

$$\mu_n = \prod_{j=1}^{n} \mathrm{I}_{\{U_{T_j} > 0\}}, \qquad \mu_0 = 1.$$

Then

$$\Gamma_{K,K} = Z(T_K) = g(U_{T_K}, t_0 - T_K)\mu_K.$$

Similarly as in Muciek (2002), Theorem 3.2, we get the following dynamic programming equations:

(i) For $n = K-1, K-2, \ldots, 0$,

$$\Gamma_{n,K} = \operatorname{ess\,sup}\Big\{\mu_n \bar{F}(\xi) g\left(U_{T_n} + \left(a + \tfrac{c}{\alpha}\right)\left(e^{\alpha(T_n+\xi)} - e^{\alpha T_n}\right) - e^{\alpha T_n}\left(e^{\alpha\xi} - 1\right)\textstyle\sum_{i=1}^{n} X_n e^{\beta_1 T_i}, t_0 - T_n - \xi\right) + \\ + \mathsf{E}(\mathrm{I}_{\{\xi \geqslant S_{n+1}\}}\Gamma_{n+1,K}|\mathcal{F}_n) : \xi \geqslant 0 \text{ is } \mathcal{F}_n\text{-measurable}\Big\} \quad \text{a.s.},$$

where $\bar{F} = 1 - F$ is the survival function.

(ii) For $n = K, K-1, \ldots, 0$,

$$\Gamma_{n,K} = \mu_n \gamma_{K-n}(U_{T_n}, T_n) \quad \text{a.s.},$$

where the sequence of functions $\{\gamma_j(u,t), u \in \mathbb{R}, t \geqslant 0\}$ is defined as follows

$$\begin{aligned}
\gamma_0(u,t) &= g(u, t_0 - t),\\
\gamma_j(u,t) &= \sup_{r \geqslant 0}\Bigg[\bar{F}(r) g\left(ue^{\alpha r} + \frac{1}{\alpha}\left(e^{\alpha(t+r)} - e^{\alpha t} + c(e^{\alpha t} - 1)(e^{\alpha r} - 1)\right), t_0 - t - r\right) + \\
&\quad + \int_0^r dF(s) \int_0^{ue^{\alpha r} + \frac{1}{\alpha}\left(e^{\alpha(t+r)} - e^{\alpha t} + c(e^{\alpha t} - 1)(e^{\alpha r} - 1)\right)} \\
&\quad \gamma_{j-1}\left(ue^{\alpha r} + \frac{1}{\alpha}\left(e^{\alpha(t+r)} - e^{\alpha t} + c(e^{\alpha t} - 1)(e^{\alpha r} - 1)\right) - xe^{\beta_1(t+s)}, t + s\right) dH(x)\Bigg],
\end{aligned}$$

$j = 1, 2, \ldots$

The above equations differ from the ones in Theorem 3.2 in Muciek (2002) as a result of a different form of the capital assets process U_t.

The next step is to find the optimal stopping time τ^*_K. To do this we should analyze the properties of the sequence of functions $\{\gamma_n, n \geqslant 0\}$.

Let $B = B[(-\infty, +\infty) \times [0, +\infty)]$ be the space of all bounded and continuous functions with the norm $||\delta|| = \sup_{u,t} |\delta(u,t)|$ and let

$$B^0 = \{\delta : \ \delta(u,t) = \delta_1(u,t)\mathrm{I}_{\{t \leqslant t_0\}} \text{ and } \delta_1 \in B\}.$$

One should notice that the functions $\{\gamma_n, n \geqslant 0\}$ are included in B^0. For each $\delta \in B^0$ and any $u \in \mathbb{R}$, $t, r \geqslant 0$ let

$$\begin{aligned}\phi_\delta(r,u,t) &= \bar{F}(r) g\left(ue^{\alpha r} + \frac{1}{\alpha}\left(e^{\alpha(t+r)} - e^{\alpha t} + c(e^{\alpha t}-1)(e^{\alpha r}-1)\right), t_0 - t - r\right) + \\ &+ \int_0^r dF(s)\Bigg[\int_0^{ue^{\alpha r}+\frac{1}{\alpha}\left(e^{\alpha(t+r)}-e^{\alpha t}+c(e^{\alpha t}-1)(e^{\alpha r}-1)\right)} \\ &\delta\left(ue^{\alpha r} + \frac{1}{\alpha}\left(e^{\alpha(t+r)} - e^{\alpha t} + c(e^{\alpha t}-1)(e^{\alpha r}-1)\right) - xe^{\beta(t+s)}, t+s\right) dH(x)\Bigg].\end{aligned}$$

From the properties of the cumulative distribution function F we know that $\phi_\delta(r,u,t)$ has at most a countable number of points of discontinuity according to r and is continuous according to (u,t) in the case of $g_1(\cdot)$ being continuous and $t \neq t_0 - r$. Therefore, for further considerations we need the following assumptions

Assumption 3.2 *The function $g_1(\cdot)$ is bounded and continuous.*

For each $\delta \in B^0$ let

$$(\Phi\delta)(u,t) = \sup_{r \geqslant 0}\{\phi_\delta(r,u,t)\}. \tag{9}$$

Lemma 3.3 *For each $\delta \in B^0$ we have*

$$(\Phi\delta)(u,t) = \max_{0 \leqslant r \leqslant t_0 - t}\{\phi_\delta(r,u,t)\} \in B^0$$

and there exists a function $r_\delta(u,t)$ such that $(\Phi\delta)(u,t) = \phi_\delta(r_\delta(u,t),u,t)$.

In subsequent considerations more properties of Φ will be presented.

For $i = 1, 2, \ldots$ and $u \in \mathbb{R}$, $t \geqslant 0$, $\gamma_i(u,t)$ may be expressed as follows

$$\gamma_i(u,t) = \begin{cases} (\Phi\gamma_{i-1})(u,t) & \text{if } u \geqslant 0 \text{ and } t \leqslant t_0, \\ 0 & \text{otherwise,} \end{cases}$$

and from Lemma 3.3 there exist functions $r_{\gamma_{i-1}}$ such that

$$\gamma_i(u,t) = \begin{cases} \phi_{\gamma_{i-1}}(r_{\gamma_{i-1}}, u, t) & \text{if } u \geqslant 0 \text{ and } t \leqslant t_0, \\ 0 & \text{otherwise.} \end{cases}$$

To specify the form of the optimal stopping times $\tau^*_{n,K}$, we need to define the following random variables

$$R^*_i = r_{\gamma_{K-i+1}}(U_{T_i}, T_i)$$

and

$$\sigma_{n,K} = K \wedge \inf\{i \geqslant n :\ R^*_i < S_{i+1}\}.$$

Finally in Corollary 3.4 we present the form of the optimal stopping time.

Corollary 3.4 *Let*

$$\tau^*_{n,K} = T_{\sigma_{n,K}} + R^*_{\sigma_{n,K}} \quad \text{and} \quad \tau^*_K = \tau^*_{0,K},$$

then for all $0 \leqslant n \leqslant K$ the following hold

$$\Gamma_{n,K} = \mathsf{E}(Z(\tau^*_{n,K})|\mathcal{F}_n) \ a.s. \quad \text{and} \quad \Gamma_{0,K} = \mathsf{E}(Z(\tau^*_K)) = \gamma_K(a,0),$$

which means $\tau^*_{n,K}$ and τ^*_K are optimal stopping times in the classes $\mathcal{T}_{n,K}$ and $\mathcal{T}_{0,K}$ respectively.

4 Case with an infinite number of claims

While $\mathcal{T}$ is the set of all stopping times with respect to the family $\{\mathcal{F}(t), t \geqslant 0\}$, we would like to maximize the mean return (5), i.e. to find the optimal stopping time τ^*, such that

$$\mathsf{E}Z(\tau^*) = \sup\{\mathsf{E}Z(\tau) : \tau \in \mathcal{T}\} \tag{10}$$

is fulfilled. It will be shown that τ^* can be defined as the limit of the finite horizon optimal stopping times.

Let τ_n^* be an optimal stopping time taken from the set of stopping times which occur not earlier then the time of n-th claim, T_n,

$$\mathsf{E}Z(\tau_n^*) = \sup\{\mathsf{E}Z(\tau) : \tau \in \mathcal{T} \cap \{\tau : \tau \geqslant T_n\}\}. \tag{11}$$

The solution of this case will be based on the iteration of the operator Φ defined by (9). By an assumption that the interarrival time is greater than t_0 with nonzero probability it can proved that the operator Φ is a contraction and it has a fixed point.

The essence of this section is contained in the following theorem. The proof is based on the proof of the existence of optimal stopping times for semi-Markov processes presented by Boshuizen & Gouweleeuw (1993).

Theorem 4.1 *Assuming the utility function g_1 is differentiable and nondecreasing and F has a density function f we have*

(i) *for $n = 0, 1, \ldots$, the limit $\tilde{\tau}_n := \lim_{K\to\infty} \tau^*_{n,K}$ exists and $\tilde{\tau}_n$ is an optimal stopping time in $\mathcal{T} \cap \{\tau : \tau \geqslant T_n\}$ ($\tau^*_{n,K}$ is a solution of the case with finite number of claims defined by (7)),*

(ii) $\mathsf{E}[Z(\tilde{\tau}_n)|\mathcal{F}_n] = \mu_n \gamma(U_{T_n}, T_n)$ *a.s.*

The optimality of $\tilde{\tau}_n$ may be proved in a similar way as in Boshuizen & Gouweleeuw (1993). The details may be found in Muciek (2002). The strong generator of $\xi_t = (t, U_t, Y_t, V_t)$, $Y_t = t - T_{N(t)}$, $V_t = \mu_{N(t)}$, $t \geqslant 0$, A has the following form

$$(A\tilde{g})(t,u,y,v) = \left\{ \left((c+1)e^{\alpha t} + \alpha u - c\right) g_1'(u) - \frac{f(y)}{\bar{F}(y)} \left[g_1(u) - \int_0^u g_1(u - e^{\beta_1 t}x) dH(x) \right] \right\} v,$$

where $t < t_0$, $y \geqslant 0$ and $v \in \{0, 1\}$.

Thus we get

$$(A\tilde{g})(\xi_s) = \left\{ e^{\alpha t}(a\alpha + c - \alpha \sum_{n=0}^{N(s)} X_n e^{\beta_1 T_n}) g_1'(U_s) + \frac{f(s - T_{N(s)})}{\bar{F}(s - T_{N(s)})} \left[\int_0^{U_s} g_1(U_s - e^{\beta_1 t}x) dH(x) - g_1(U_s) \right] \right\} \mu_{N(s)}.$$

It should also be marked that the limit of optimal stopping times as $K \to \infty$ coincide with the overall optimal stopping time.

The results presented above are the modifications of the results presented in Muciek (2002). They form a step in adjusting the model to the reality. In fact the form of optimal stopping remained the same, but some calculations have changed. To get much closer to the reality some other assumptions should be weakened. For example the times between losses may not be identically distributed as in real life they are not. We are now working on such generalizations and they should appear soon.

I would like to state special thanks to prof. Krzysztof Szajowski for constructive discussions and showing further possibilities of the model development.

References

[1] Boshuizen F. A. Gouweleeuw J. M. (1993), *General optimal stopping theorems for semi-Markov processes*, Adv. in Appl. Probab., 4, 825-846.

[2] Davis M. (1976), *The representation of martingales of jump processes*, SIAM J. Control Optim., 14, 623-638.

[3] Ferenstein E.Z., Sierociński A. (1997), *Optimal Stopping of a Risk Process*, Applicationes Mathematicae, 24,3, 335-342.

[4] Jensen U. (1997), *An optimal stopping problem in risk theory*, Scand. Actuarial J.,2 149-159.

[5] Muciek B. (2002), *Optimal stopping of a risk process: model with interest rates*, J. Appl. Prob., 39, 2, 261-270.

[6] Rolski T., Schmidli H., Schimdt V., Teugels J. (1998), *Stochastic Processes for Insurance and Finance* (Wiley, Chichester)

[7] Schöttl A. (1998), *Optimal stopping of a risk reserve process with interest and cost rates*, J. Appl. Prob., XX, 115-123.