Selection of the Shrinkage Factor for the Two Stage Testimator of the Normal Mean Using Bootstrap Likelihood

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Abstract

In this paper, a new methodology based on the likelihood of bootstrap samples is introduced for improving the efficiency of the two-stage shrinkage testimator of Waikar et al.(2001). In particular, this method is useful for selecting the shrinkage factor for the two-stage testimator in view of increasing the efficiency of such testimator. The method is useful for number of estimation problem. However, in this paper, only the estimation of the normal mean is considered for related simulation studies and the discussion.

1. Introduction

The methodology of this paper is based on a concept of two stage shrinkage testimator [Waikar et al. (1984)]. This concept and its recent bootstrap version [Waikar et al. (2001)] are described below in section 1.1 for ready reference. For brevity, we provide only a short discussion.The main results of this paper, the two stage testimators for the normal mean based on the likelihoods of bootstrap samples, are presented in section 2. The efficiencies of these new testimators are studied using the simulations. These results and conclusions are given in section 3.

1.1. Two Stage Shrinkage Testimator [Waikar et al. (1984),Waikar et al.(2001)]

Let X ~ Normal (μ, σ). Suppose σ is known. Then the concept of two stage testimator for estimating μ is as follows:

Select a first stage sample of size n_1 on X. Let $\overline{X}_1$ be the sample mean.
Test the hypothesis Ho: $\mu = \mu_0$ against a two sided alternative Ha at level α using normal test. If Ho is accepted the testimator for μ is defined as

$$\hat{\mu}_w = k\overline{X}_1 + (1-k)\mu_0 \text{, where } k = \sqrt{n_1}\,| \overline{X}_1 - \mu_0|/(z_{\alpha/2}\sigma) , \tag{1}$$

where k is the shrinkage factor.
If Ho is rejected the second sample of size n_2 is selected and the testimator is defined as

$$\widehat{\mu_w} = (n_1\overline{X}_1 + n_2\overline{X}_2)/(n_1+n_2) , \tag{2}$$

where $\overline{X}_2$ is the mean of the second sample.
The efficiency of $\hat{\mu}_w$ as compared with usual estimator of μ based on the equivalent sample size is studied in Waikar et al. (1984).
Instead of using k as defined in (1), Waikar et al.(2001) have considered the use of bootstrap samples from the first stage sample for generating a sequence of values of the shrinkage factor k denoted by k*, where $k^* = \sqrt{n}_1 | \overline{X}_1^* - \mu_0|/(z_{\alpha/2}\sigma)$ and $\overline{X}_1^*$ is the mean of a bootstrap sample. Then, B bootstrap samples are selected and the mean $\overline{k\,*}$ of B values of $k*$ is calculated. Now, in (1) the shrinkage factor k is replaced by $\overline{k\,*}$. This testimator is denoted by $\hat{\mu}_r$. The second stage testimator remains the same. Further, Waikar et al. (2001) have used the simulations to show that $\hat{\mu}_r$ is more efficient than $\widehat{\mu_w}$.
The modifications of the testimator $\hat{\mu}_w$ based on likelihood are given below is section 2.

2. Selection of shrinkage factor using Likelihoods for Bootstrap Samples

In this section, we introduce a new procedure, based on the likelihoods of bootstrap samples, for generating a sequence of values of k and then selecting a value of k in $\widehat{\mu_w}$. Let $X_1, X_2, \ldots, X_{n_1}$ be the first stage sample, as mentioned in the above Waikar et al.(1984) method, with sample mean $\overline{X}_1$. The form of the testimator (1) suggests that ; (a) $\overline{X}_1$ and μ_0 jointly provide a prior information that can be used for the estimation of μ, and (b) the normal mean is of the form

$$\mu = k\overline{X}_1 + (1-k)\mu_0 \, , \tag{3}$$

where k can be interpreted as a parameter to be determined subject to the definition of k $(k = \sqrt{n_1}\,|\,\overline{X}_1 - \mu_0|/(z_{\alpha/2}\sigma))$. Thus, the estimation of k leads us to the estimator of μ. However, we note that the estimation of k (or selecting k) using the classical or traditional methods does not make sense due to its dependence on n_1, $\overline{X}_1$, μ_0, σ, and α. Further, we note that it is difficult to predict which of these quantities will have more influence than others on k. Therefore, we need different criteria that can incorporate the influence of these quantities on the estimation of k, and also provide us a method for estimating k. Again , we note that the only information available for exploring such influence is through the first stage sample of size n_1 on X. Therefore, exploring the log-likelihood of the normal distribution (with (3) as its mean) based on the first stage sample or the bootstrap samples selected from the first stage sample seems to be a logical step. The formal procedure is outlined below. In particular, we consider the normal likelihood functions for bootstrap samples as follows :
Let X have a normal (μ, σ) distribution. Then, for a given first stage sample of size n_1 the log-likelihood function $L(\mu, \sigma)$ is given by

$$L(\mu, \sigma) = -n_1 \, ln\,(\sigma\sqrt{2\pi}) - \sum_{i=1}^{n_1}[(x_i - \mu)^2/(2\,\sigma^2)] \tag{4}$$

Further, as mentioned above, let μ be as in (3). Then, the log-likelihood function (4) when Ho is not rejected, is given by

$$L(\mu, \sigma) = -n_1 \, ln\,(\sigma\sqrt{2\pi}) - \sum_{i=1}^{n_1}(x_i - k\overline{X}_1 - (1-k)\mu_0)^2/(2\,\sigma^2). \tag{5}$$

where $k = \sqrt{n_1}\,|\,\overline{X}_1 - \mu_0|/(z_{\alpha/2}\sigma)$ is the same as in Waikar et al.(1984, 2001).

Now, let $X_1{*}, X_2{*}, \ldots, X_{n_1}{*}$ be the bootstrap sample from $X_1, X_2, \ldots, X_{n_1}$.
Let $\overline{X_1}^{*}$ denote the corresponding mean of the bootstrap sample.
Now, the formal procedure for obtaining the testimator using (5) is as follows:
First, we redefine (5) in four different ways given below in (6)-(9), and then use these log-likelihoods for defining four new two stage shrinkage testimators of μ.

Let

$$L_1 = -n_1 ln(\sqrt{2\pi}\;\;\sigma) - \tfrac{1}{2\sigma^2}\sum(x_i^* - k^*\overline{X_1}^{*} - (1-k^*)\mu_0)^2 \;\;, \tag{6}$$

$$L_2 = -n_1 ln(\sqrt{2\pi}\;\;\sigma) - \tfrac{1}{2\sigma^2}\sum(x_i^* - k^*\overline{X_1} - (1-k^*)\mu_0)^2, \tag{7}$$

$$L_3 = -n_1 ln(\sqrt{2\pi}\;\;\sigma) - \tfrac{1}{2\sigma^2}\sum(x_i - k^*\overline{X_1^*} - (1-k^*)\mu_0)^2, \tag{8}$$

$$L_4 = -n_1 ln(\sqrt{2\pi}\;\;\sigma) - \tfrac{1}{2\sigma^2}\sum(x_i - k^*\overline{X_1} - (1-k^*)\mu_0)^2. \tag{9}$$

Suppose that B bootstrap samples are drawn from the first stage sample. For each sample find k^*using the formula $k^* = \sqrt{n_1}\,|\,\overline{X_1}^{*} - \mu_0|/(z_{\alpha/2}\sigma)$and evaluate (6).Thus, we have B values of log-likelihood function given by (6). From these B values of log-likelihood find the one with maximum value of L_1. Let the corresponding k^* be denoted by k^*_{L1} .The corresponding testimator of μ is now defined as

$$\widehat{\mu_{L1}} = k^*_{L1}\overline{X}_1 + (1 - k^*_{L1})\mu_0 \text{, when Ho is not rejected, and}$$
$$\widehat{\mu_{L1}} = (n_1\overline{X}_1 + n_2\overline{X}_2)/(n_1 + n_2) \text{, when Ho is rejected.} \qquad (10a)$$

Now, we repeat the above bootstrap procedure for the log-likelihoods in (7)-(9) and obtain the three other testimators listed below, where k^*_{L2}, k^*_{L3} , k^*_{L4} are the corresponding shrinkage factors.

$$\widehat{\mu_{L2}} = k^*_{L2}\overline{X}_1 + (1 - k^*_{L2})\mu_0 \text{, when Ho is not rejected, and}$$
$$\widehat{\mu_{L2}} = (n_1\overline{X}_1 + n_2\overline{X}_2)/(n_1 + n_2) \text{, when Ho is rejected.} \qquad (10b)$$

$$\widehat{\mu_{L3}} = k^*_{L3}\overline{X}_1 + (1 - k^*_{L3})\mu_0 \text{, when Ho is not rejected, and}$$
$$\widehat{\mu_{L3}} = (n_1\overline{X}_1 + n_2\overline{X}_2)/(n_1 + n_2) \text{, when Ho is rejected.} \qquad (10c)$$

$$\widehat{\mu_{L4}} = k^*_{L4}\overline{X}_1 + (1 - k^*_{L4})\mu_0 \text{, when Ho is not rejected, and}$$
$$\widehat{\mu_{L4}} = (n_1\overline{X}_1 + n_2\overline{X}_2)/(n_1 + n_2) \text{, when Ho is rejected.} \qquad (10d)$$

Remark1. Since we are drawing a finite number B of bootstrap sample, sometimes more than one value of k^*'s used in defining (10a) -(10d) may give the same maximum value or respective log-likelihoods. In such cases, we suggest that,(1) either increase the number B of bootstrap samples to be drawn until unique value is obtained or ,(2)choose the minimum of such k^*'s.The second choice is based on the fact that since we are accepting Ho such choice of k^* should give more weightage to μ_0.

Remark 2. In the above derivations we have assumed that the normal standard deviation σ is known. For the case where σ is unknown the above derivations are repeated using the Student's t-test instead of the normal test.

Remark 3. The two stage testimators for the mean of the exponential distribution using the likelihood can be derived using the above argument and an appropriate hypothesis testing procedure for the exponential mean. The results for this case are not included here for the sake of brevity.

3. The Efficiency of the Estimators

In Waikar et al.(2001), as mentioned earlier, it is shown that $\hat{\mu}_r$ is more efficient than $\hat{\mu}_w$ for different values of μ ,σ and α, and (n_1, n_2). Now, we compare the efficiencies of our new tesimators $\widehat{\mu_{L1}}$, $\widehat{\mu_{L2}}$, $\widehat{\mu_{L3}}$, and $\widehat{\mu_{L4}}$ with the respective efficiencies of $\hat{\mu}_w$ and $\hat{\mu}_r$. The simulation results are listed below. These results are based on 200 bootstrap samples drawn from simulated samples from normal distributions.

The following table is the display of efficiency calculations for one of the several cases of simulations used in this study. For simulations,we used different sample sizes and different values of μ_0, σ, and α.

Table 0. The Calculations for Efficiencies of Six Testimators - A Typical Simulation Example

$n_1 =$ 10 , $n_2 =$ 10 , $\mu =$ 10 μ_0= 10.0 α = 0.10

$\frac{MSE(\hat{\mu}_w)}{MSE(\hat{\mu}_{L1})}$	$\frac{MSE(\hat{\mu}_r)}{MSE(\hat{\mu}_{L1})}$	$\frac{MSE(\hat{\mu}_w)}{MSE(\hat{\mu}_{L2})}$	$\frac{MSE(\hat{\mu}_r)}{MSE(\hat{\mu}_{L2})}$	$\frac{MSE(\hat{\mu}_w)}{MSE(\hat{\mu}_{L3})}$	$\frac{MSE(\hat{\mu}_r)}{MSE(\hat{\mu}_{L3})}$	$\frac{MSE(\hat{\mu}_w)}{MSE(\hat{\mu}_{L4})}$	$\frac{MSE(\hat{\mu}_r)}{MSE(\hat{\mu}_{L4})}$	σ
1.015	0.733	1.226	0.885	0.805	0.581	0.805	0.581	1
1.086	0.693	1.520	0.971	0.785	0.502	0.785	0.501	2
1.141	0.695	1.427	0.870	0.816	0.497	0.816	0.497	3
1.161	0.794	1.089	0.745	0.783	0.535	0.781	0.534	4
1.151	0.776	1.556	1.050	0.829	0.559	0.830	0.560	5
1.115	0.805	1.305	0.942	0.809	0.584	0.808	0.584	6
1.133	0.760	1.510	1.013	0.809	0.542	0.809	0.543	7
1.136	0.696	1.412	0.865	0.816	0.500	0.817	0.500	8
1.119	0.827	1.277	0.943	0.846	0.624	0.845	0.624	9

$n_1 =$ 10 , $n_2 =$ 10 , $\mu =$ 10 μ_0= 10.0 α = 0.05

$\frac{MSE(\hat{\mu}_w)}{MSE(\hat{\mu}_{L1})}$	$\frac{MSE(\hat{\mu}_r)}{MSE(\hat{\mu}_{L1})}$	$\frac{MSE(\hat{\mu}_w)}{MSE(\hat{\mu}_{L2})}$	$\frac{MSE(\hat{\mu}_r)}{MSE(\hat{\mu}_{L2})}$	$\frac{MSE(\hat{\mu}_w)}{MSE(\hat{\mu}_{L3})}$	$\frac{MSE(\hat{\mu}_r)}{MSE(\hat{\mu}_{L3})}$	$\frac{MSE(\hat{\mu}_w)}{MSE(\hat{\mu}_{L4})}$	$\frac{MSE(\hat{\mu}_r)}{MSE(\hat{\mu}_{L4})}$	σ
1.019	0.741	1.190	0.865	0.813	0.591	0.815	0.592	1
1.091	0.741	1.244	0.845	0.771	0.524	0.770	0.523	2
1.121	0.877	1.484	1.162	0.794	0.622	0.793	0.620	3
1.171	0.823	1.437	1.010	0.770	0.541	0.771	0.542	4
0.927	0.572	1.777	1.096	0.698	0.431	0.698	0.431	5
1.268	0.704	1.566	0.869	0.731	0.406	0.733	0.407	6
1.036	0.771	1.361	1.013	0.713	0.531	0.710	0.529	7
1.012	0.727	1.343	0.965	0.750	0.539	0.749	0.538	8
1.435	0.871	1.934	1.174	0.764	0.464	0.766	0.465	9

$n_1 =$ 10 , $n_2 =$ 10 , $\mu =$ 10 μ_0= 10.0 α = 0.01

$\frac{MSE(\hat{\mu}_w)}{MSE(\hat{\mu}_{L1})}$	$\frac{MSE(\hat{\mu}_r)}{MSE(\hat{\mu}_{L1})}$	$\frac{MSE(\hat{\mu}_w)}{MSE(\hat{\mu}_{L2})}$	$\frac{MSE(\hat{\mu}_r)}{MSE(\hat{\mu}_{L2})}$	$\frac{MSE(\hat{\mu}_w)}{MSE(\hat{\mu}_{L3})}$	$\frac{MSE(\hat{\mu}_r)}{MSE(\hat{\mu}_{L3})}$	$\frac{MSE(\hat{\mu}_w)}{MSE(\hat{\mu}_{L4})}$	$\frac{MSE(\hat{\mu}_r)}{MSE(\hat{\mu}_{L4})}$	σ
0.868	0.678	1.808	1.414	0.642	0.502	0.643	0.503	1
0.842	0.688	1.435	1.173	0.596	0.487	0.596	0.487	2
1.133	0.888	2.261	1.772	0.713	0.559	0.712	0.558	3
1.170	0.869	1.845	1.371	0.653	0.485	0.649	0.482	4
0.679	0.523	1.243	0.956	0.626	0.481	0.626	0.481	5
1.089	0.851	1.676	1.309	0.675	0.528	0.675	0.527	6
1.102	0.871	1.562	1.235	0.634	0.501	0.635	0.502	7
0.992	0.720	1.277	0.926	0.689	0.500	0.691	0.502	8
1.286	0.951	1.577	1.166	0.740	0.547	0.739	0.547	9

Some of the conclusions drawn from such simulations are given below in Tables 1, 2, 3.

3.1. Comparison of the efficiency of $\widehat{\mu}_w$ and $\hat{\mu}_r$ with those of $\widehat{\mu}_{Li}$, $i = 1, 2, 3, 4$.

The simulated samples are from the normal distribution with μ=10, and σ =1 , 2. (Other values of sigma were studied. For brevity the results for only above values are recorded.).

First, in Tables .1a and 1b ,we study the efficiencies of the testimators when μ_0 approaches the true value μ=10 form below when α=0.1.(Other values of α were studied. Due to the similarity in the results here we report results for α=0.1.). The case where μ_0 approaches from μ=10 above was aslo studied.

Table 1a: Comparison of the efficiency of $\widehat{\mu}_w$ with $\widehat{\mu}_{Li}$, $i = 1, 2, 3, 4$.

μ_0	σ	$\frac{MSE(\widehat{\mu}_w)}{MSE(\widehat{\mu}_{L1})}$	$\frac{MSE(\widehat{\mu}_w)}{MSE(\widehat{\mu}_{L2})}$	$\frac{MSE(\widehat{\mu}_w)}{MSE(\widehat{\mu}_{L3})}$	$\frac{MSE(\widehat{\mu}_w)}{MSE(\widehat{\mu}_{L4})}$
9.5	1	0.854	0.841	1.059	1.058
	2	0.967	0.882	0.979	0.912
9.7	1	1.091	0.954	0.986	0.988
	2	0.962	1.104	0.883	0.883
9.9	1	1.082	1.287	0.866	0.867
	2	1.257	1.356	0.804	0.804
10.0	1	1.015	1.226	0.805	0.805
	2	1.086	1.520	0.785	0.785

Table 1b: Comparison of the efficiency of $\widehat{\mu}_r$ with $\widehat{\mu}_{Li}$, $i = 1, 2, 3, 4$.

μ_0	σ	$\frac{MSE(\widehat{\mu}_r)}{MSE(\widehat{\mu}_{L1})}$	$\frac{MSE(\widehat{\mu}_r)}{MSE(\widehat{\mu}_{L2})}$	$\frac{MSE(\widehat{\mu}_r)}{MSE(\widehat{\mu}_{L3})}$	$\frac{MSE(\widehat{\mu}_r)}{MSE(\widehat{\mu}_{L4})}$
9.5	1	0.963	0.949	1.194	1.193
	2	0.902	0.822	0.913	0.912
9.7	1	1.017	0.890	0.920	0.921
	2	0.780	0.896	0.716	0.717
9.9	1	0.800	0.951	0.640	0.641
	2	0.760	0.804	0.804	0.486
10.0	1	0.733	0.885	0.581	0.581
	2	0.693	0.971	0.502	0.501

Conclusions.

The Table 1a results lead to the following conclusions.

1. The testimators $\widehat{\mu}_{L1}$ and $\widehat{\mu}_{L2}$ are, in general, more efficient than $\widehat{\mu}_w$ as μ_0 approaches true mean of 10. Further, $\widehat{\mu}_{L2}$ is more efficient than $\widehat{\mu}_{L1}$.

2. The testimators $\widehat{\mu}_{L3}$ and $\widehat{\mu}_{L4}$ are more efficient than $\widehat{\mu}_w$ when μ_0 =9.5 or less (or μ_0 =10.5 or more), that is when μ_0 is farther away from true value of μ=10 .

3. The observations in (1) and (2) above are useful for increasing the efficiency of the two stage shrinkage testimators using the information obtained through testing H_0.

The Table 1b leads to the following conclusions.

4. As in (2) above $\widehat{\mu}_{L3}$ and $\widehat{\mu}_{L4}$ are more efficient than $\widehat{\mu}_r$ when μ_0 =9.5 or less. That is when μ_0 is farther away from true value of μ=10.

5. In general, $\widehat{\mu}_r$ is more efficient than the four new testimators except for the above conclusion in (4).

3.2. Comparison of the efficiencies of six testimators for different sample sizes.

As mentioned above the sample size has an effect on the efficiency of different testimators.
Tables 2a and 2b provide comparison of the efficiencies of six testimators for different sample sizes.

Table 2a.Comparison of efficiency of $\hat{\mu}_w$ and $\hat{\mu}_{Li}$, $i = 1, 2, 3, 4$ for different sample sizes.

n_1	n_2	σ	$\frac{MSE(\widehat{\mu}_w)}{MSE(\widehat{\mu}_{L1})}$	$\frac{MSE(\widehat{\mu}_w)}{MSE(\widehat{\mu}_{L2})}$	$\frac{MSE(\widehat{\mu}_w)}{MSE(\widehat{\mu}_{L3})}$	$\frac{MSE(\widehat{\mu}_w)}{MSE(\widehat{\mu}_{L4})}$
10	10	1	0.854	0.841	1.059	1.058
		2	0.967	0.882	0.979	0.978
20	20	1	0.856	0.760	1.104	1.103
		2	0.845	0.919	1.014	1.016
30	30	1	0.835	0.744	1.067	1.067
		2	0.924	0.937	1.086	1.086
50	50	1	0.894	0.855	1.011	1.011
		2	0.816	0.773	1.140	1.362
100	100	1	1.000	1.000	1.000	1.000
		2	0.861	0.880	1.083	1.083

Table 2b.Comparison of efficiency of $\widehat{\mu}_r$ and $\hat{\mu}_{Li}$, $i = 1, 2, 3, 4$ for different sample sizes.

n_1	n_2	σ	$\frac{MSE(\widehat{\mu}_r)}{MSE(\widehat{\mu}_{L1})}$	$\frac{MSE(\widehat{\mu}_r)}{MSE(\widehat{\mu}_{L2})}$	$\frac{MSE(\widehat{\mu}_r)}{MSE(\widehat{\mu}_{L3})}$	$\frac{MSE(\widehat{\mu}_r)}{MSE(\widehat{\mu}_{L4})}$
10	10	1	0.963	0.949	1.194	1.193
		2	0.902	0.822	0.913	0.912
20	20	1	1.050	0.933	1.356	1.354
		2	0.878	0.956	1.054	1.056
30	30	1	0.999	0.890	1.276	1.277
		2	0.962	0.975	1.130	1.130
50	50	1	0.965	0.923	1.092	1.092
		2	0.974	0.922	1.361	1.362
100	100	1	1.000	1.000	1.000	1.000
		2	1.025	1.047	1.289	1.288

Conclusions.

1. In general, the efficiencies of $\hat{\mu}_{L3}$, $\hat{\mu}_{L4}$ are higher than those of $\widehat{\mu}_w$ and $\widehat{\mu}_r$ for different sample sizes.

2. In general, the efficiencies of $\hat{\mu}_{L1}$, $\hat{\mu}_{L2}$ are lower than those of $\widehat{\mu}_w$ and $\widehat{\mu}_r$ for different sample sizes.

3. For $n_1 = n_2 = 100$ and σ=1 all the six testimators are equally efficient.

3.3. Effect of the level of significance α on the efficiencies of $\hat{\mu}_{Li}$, $i = 1,\ 2,\ 3, 4$.

As mentioned earlier the likelihood function (5) also depends on the level of significance α through $z_{\alpha/2}$ and the decision about the rejection of μ_0.Therefore, the study of changes in efficiencies with values of α are considered in this paper. In what follow, we provide one such situation as an example. The other cases can be srtudied in similar way.

The efficiencies $\hat{\mu}_{Li}$, $i = 1,\ 2,\ 3, 4$ from the simulation results for $n_1 =$ 100 , $n_2 =$ 100 , μ=10, $\mu_0 = 9.5$, σ=1,2 , and for three levels of α = 10% , 5%, 1% are shown below in Tables 3a and 3b.

Table 3a. Comparison of efficiencies of $\hat{\mu}_w$ and $\hat{\mu}_{Li}$, $i = 1,\ 2,\ 3, 4$ for different values of α and σ

α	σ	$\frac{MSE(\widehat{\mu}_w)}{MSE(\widehat{\mu}_{L1})}$	$\frac{MSE(\widehat{\mu}_w)}{MSE(\widehat{\mu}_{L2})}$	$\frac{MSE(\widehat{\mu}_w)}{MSE(\widehat{\mu}_{L3})}$	$\frac{MSE(\widehat{\mu}_w)}{MSE(\widehat{\mu}_{L4})}$
10%	1	1.000	1.000	1.000	1.000
	2	0.861	0.880	1.083	1.083
5%	1	1.000	1.000	1.000	1.000
	2	0.673	0.722	1.132	1.129
1%	1	1.000	1.000	1.000	1.000
	2	0.810	0.822	1.155	1.154

Table 3b. Comparison of efficiencies of $\hat{\mu}_r$ and $\widehat{\mu}_{Li}$ $i = 1,\ 2,\ 3, 4$ for different values of α and σ

α	σ	$\frac{MSE(\widehat{\mu}_r)}{MSE(\widehat{\mu}_{L1})}$	$\frac{MSE(\widehat{\mu}_r)}{MSE(\widehat{\mu}_{L2})}$	$\frac{MSE(\widehat{\mu}_r)}{MSE(\widehat{\mu}_{L3})}$	$\frac{MSE(\widehat{\mu}_r)}{MSE(\widehat{\mu}_{L4})}$
10%	1	1.000	1.000	1.000	1.000
	2	1.025	1.047	1.289	1.288
5%	1	1.000	1.000	1.000	1.000
	2	0.902	0.969	1.517	1.514
1%	1	1.000	1.000	1.000	1.000
	2	0.909	0.923	1.296	1.295

Conclusions:

1. In Table 3a as α increases the efficiencies of $\widehat{\mu}_{L3}$ and $\widehat{\mu}_{L4}$ increase steadily.

2. In Tables 3b as α increases the efficiencies of $\widehat{\mu}_{L3}$ and $\widehat{\mu}_{L4}$ first increase and then decrease.

4. Summary

In this paper, we introduced four two stage shrinkage testimators. These testimators are obtained using the bootstrap samples drawn from the first stage sample. The efficiencies of these testimators were studied using the simulations. The discussion regarding the usefulness of these testimators under different situations is provided as conclusions from various tables obtained from simulation results.

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